Frobenius manifold via Weyl group invariant theory

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Roughly speaking,

$$\text{Frobenius manifold} \quad = \quad \text{complex manifold} \quad + \quad \text{Frobenius structure}$$
$$(\fallingdotseq \text{flat structure})$$

There exist 3 different constructions of Frobenius manifolds:

(GW) genus 0 Gromov–Witten theory,

(Def.) deformation theory + primitive form,

(Weyl) root system + Weyl group invariant theory.

Classical Mirror Symmetry

A Frobenius manifold constructed by (GW) (resp. (Def.)) is isomorphic to one constructed by (Def.) (resp. (GW)).

On the other hand, the construction (Weyl) is related with (Def.) by the period mapping of a (certain) primitive form.

Example: ADE

(Def.)		(Weyl)
ADE singularity	$\longleftrightarrow$	ADE root system
$f : \mathbb{C}^3 \longrightarrow \mathbb{C}$		$(L, I, \Delta^{\mathrm{re}})$

The isomorphism of Frobenius manifolds between (Def.) and (Weyl) is induced by the period mapping of the primitive form $\zeta = dz_1 \wedge dz_2 \wedge dz_3$.

In order to see global description of the period mapping of a primitive form, Frobenius manifolds via (Weyl) play an important role.

The construction (Weyl) is known a few cases;

- finite Weyl group
  [Saito, Saito–Yano–Sekiguchi, Dubrovin],
- extended affine Weyl group
  [Dubrovin–Zhang, Dubrovin–Zhang–Zuo, Zuo],
- elliptic Weyl group
  [Saito, Satake, Dubrovin, Bertola],

$\vdots$

Problem

Establish a construction of Frobenius structures by the invariant theory of the Weyl group for a given (generalized) root system.

Let $\mathcal{D}$ be a $\mathbb{C}$-linear triangulated category and $K_0(\mathcal{D})$ the Grothendieck group of $\mathcal{D}$.

A stability condition $(Z, \mathcal{P})$ on $\mathcal{D}$ consists of

- $Z : K_0(\mathcal{D}) \longrightarrow \mathbb{C}$; group homomorphism (called a central charge),
- $\mathcal{P}(\phi)$: additive full sub categories ($\phi \in \mathbb{R}$),

satisfying some axioms.

Denote by $\mathrm{Stab}(\mathcal{D})$ the space of all stability conditions on $\mathcal{D}$. It is known that $\mathrm{Stab}(\mathcal{D})$ has natural topology.

Theorem 1 (Bridgeland).

The natural forgetful map

$$\mathcal{Z} : \mathrm{Stab}(\mathcal{D}) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(K_0(\mathcal{D}), \mathbb{C}), \quad (Z, \mathcal{P}) \mapsto Z,$$

is a local homeomorphism. In particular, $\mathrm{Stab}(\mathcal{D})$ *has a structure of complex manifolds.*

In the viewpoint of mirror symmetry, Takahashi conjectured the following:

Conjecture 2 (Takahashi).

Let

- $\vec{\Delta}$ *be a Dynkin quiver,* $\mathcal{D}^b(\vec{\Delta}) := \mathcal{D}^b \mathrm{mod}\, \mathbb{C}\vec{\Delta}$,
- $f : \mathbb{C}^3 \longrightarrow \mathbb{C}$ *the ADE singularity corresponding to* $\vec{\Delta}$, *and*
- $F : \mathbb{C}^3 \times M \longrightarrow \mathbb{C}$ *the universal unfolding of* f *(*$M = \mathbb{C}^n$*).*

There should exist a biholomorphic map

$$\mathrm{Stab}(\mathcal{D}^b(\vec{\Delta})) \cong M.$$

In particular, $\mathrm{Stab}(\mathcal{D}^b(\vec{\Delta}))$ *has a Frobenius structure (and real structure) induced by the Frobenius manifold* M *(with real structure) constructed by the deformation theory and primitive forms.*

[Bridgeland–Qiu–Surtherland] : A_2 case.
[Haiden–Katzarkov–Kontsevich] : A_n and affine-$A_{p,q}$ cases.

Moreover, they showed that the natural map

$$\mathcal{Z} : \mathrm{Stab}(\mathcal{D}) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(K_0(\mathcal{D}), \mathbb{C})$$

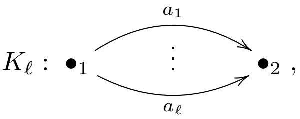
is corresponding to the exponential period mapping of a (certain) primitive form under the biholomorphic map. Namely, the following diagram cummutes:

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow{\quad\cong\quad} & \mathrm{Stab}(\mathcal{D}) \\ & {\scriptstyle \int e^F \zeta} \searrow \quad \circlearrowleft \quad \swarrow {\scriptstyle \mathcal{Z}} & \\ & \mathrm{Hom}_{\mathbb{Z}}(K_0(\mathcal{D}), \mathbb{C}) & \end{array},$$

where $\widetilde{M}$ is the universal covering space of M and ζ is a certain primitive form.

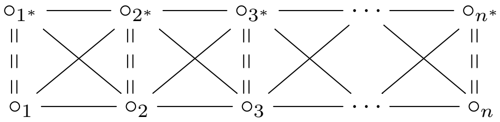
Based on the viewpoint of stability conditions, we construct Frobenius manifolds for

1. ℓ-Kronecker quiver



   The root system (Kac–Moody Lie algebra) associated with K_ℓ is of indefinite type.

2. "n-extended" affine A_n case. The Coxeter–Dynkin diagram is given by



Recall the definition of Frobenius manifold.

Let M be an n-dimensional complex manifold.

Definition 3.

Let $d \in \mathbb{C}$ and

- $\eta : \mathcal{T}_M \times \mathcal{T}_M \longrightarrow \mathcal{O}_M$*: non-degenerate symmetric* $\mathcal{O}_M$*-bilinear form,*
- $\circ : \mathcal{T}_M \times \mathcal{T}_M \longrightarrow \mathcal{T}_M$*: associative commutative* $\mathcal{O}_M$*-bilinear product,*
- $e \in \Gamma(M, \mathcal{T}_M)$*: the unit of* $\circ$*,*
- $E \in \Gamma(M, \mathcal{T}_M)$*, which is called the Euler vector field.*

The tuple $(\eta, \circ, e, E)$ is a Frobenius structure of (conformal) dimension d on M if it satisfies the following axioms:

Axioms of Frobenius structure

For any $\delta, \delta', \delta'' \in \mathcal{T}_M$,

1. The Levi-Civita connection $\nabla\!\!\!/ : \mathcal{T}_M \longrightarrow \mathcal{T}_M \otimes \Omega^1_M$ with respect to η is flat,
2. The product $\circ$ is self-adjoint with respect to η: that is,
$$\eta(\delta \circ \delta', \delta'') = \eta(\delta, \delta' \circ \delta''),$$
3. The tensor $C : \mathcal{T}_M \longrightarrow \mathrm{End}_{\mathcal{O}_M} \mathcal{T}_M$ defined by $C_\delta \delta' := \delta \circ \delta'$ is $\nabla\!\!\!/$-flat,
4. The unit element e is $\nabla\!\!\!/$-flat,
5. The product $\circ$ and the metric η are homogeneous of degree 1 and $2-d$, respectively, with respect to the Lie derivative Lie_E of the Euler vector field E: that is,
$$\mathrm{Lie}_E(\circ) = \circ, \quad \mathrm{Lie}_E(\eta) = (2-d)\eta.$$

A manifold M with a Frobenius structure $(\eta, \circ, e, E)$ is called a Frobenius manifold.

Let $(M, \eta, \circ, e, E)$ be a Frobenius manifold of dimension d.

Proposition 4.

There exists a local coordinate system $(t_1, \cdots, t_n)$ and a holomorphic function $\mathcal{F} \in \mathcal{O}_M$ such that

- $e = \partial_1, \ \ \mathrm{Ker}\nabla\!\!\!/ \cong \bigoplus_{i=1}^{n} \mathbb{C}_M \cdot \partial_i$
- η *naturally gives a* $\mathbb{C}_M$*-bilinear* $\eta : \mathrm{Ker}\nabla\!\!\!/ \times \mathrm{Ker}\nabla\!\!\!/ \longrightarrow \mathbb{C}_M$,
- $E = \sum_{i=1}^{n} \left[(1 - q_i)t_i + c_i\right] \partial_i$, *if* $q_i \neq 1$ *then* $c_i = 0$,
- $\eta(\partial_i \circ \partial_j, \partial_k) = \partial_i \partial_j \partial_k \mathcal{F}$,
- $E\mathcal{F} = (3 - d)\mathcal{F} + ($*quadratic terms in* $t_2, \cdots, t_n)$,

where $\partial_i = \dfrac{\partial}{\partial t_i}$.

The coordinate system $(t_1, \cdots, t_n)$ is called a flat coordinate system, and the function $\mathcal{F}$ is called the Frobenius potential.

$$\underset{(\eta, \circ, e, E)}{\text{Frobenius str.}} \overset{\text{local}}{=} \underset{(t_1, \ldots, t_n)}{\text{flat coordinate}} + \underset{\mathcal{F}}{\text{Frobenius potential}}$$

Define a subset $D \subset M$, called the discriminant, by

$$D := \{p \in M \mid \det(C_E)(p) = 0\}.$$

Set $M^{\mathrm{reg}} := M \setminus D$.

Definition 5.

Define a symmetric $\mathcal{O}_{M^{\mathrm{reg}}}$-bilinear form $g : \mathcal{T}_{M^{\mathrm{reg}}} \times \mathcal{T}_{M^{\mathrm{reg}}} \longrightarrow \mathcal{O}_{M^{\mathrm{reg}}}$ by

$$g(\delta, \delta') := \eta(C_E^{-1}\delta, \delta').$$

It induces a symmetric $\mathcal{O}_{M^{\mathrm{reg}}}$-bilinear form on $\Omega^1_{M^{\mathrm{reg}}}$. We call this symmetric $\mathcal{O}_{M^{\mathrm{reg}}}$-bilinear form $g : \Omega^1_{M^{\mathrm{reg}}} \times \Omega^1_{M^{\mathrm{reg}}} \longrightarrow \mathcal{O}_{M^{\mathrm{reg}}}$ the intersection form of the Frobenius manifold.

On the flat coordinate system $(t_1, \cdots, t_n)$, the intersection form g is given by

$$g(dt_i, dt_j) = \sum_{a,b=1}^{n} \eta^{ia}\eta^{jb} E\partial_a\partial_b\mathcal{F}, \quad \eta^{ia} := \eta(dt_i, dt_a).$$

Denote by ∇ the Levi–Civita connection with respect to g. The connection ∇ is called the second structure connection of the Frobenius manifold $(M, \eta, \circ, e, E)$. It is known that the connection ∇ is flat.

Definition 6.

Define a local system $\mathrm{Sol}(\nabla)$ *on* M^{reg} *by*

$$\mathrm{Sol}(\nabla) := \{x \in \mathcal{O}_{M^{\mathrm{reg}}} \,|\, \nabla dx = 0\}.$$

Fix a point $p_0 \in M^{\mathrm{reg}}$. Then one can obtain a group homomorphism

$$\pi_1(M^{\mathrm{reg}}, p_0) \longrightarrow \mathrm{Aut}(\mathrm{Sol}(\nabla)_{p_0}).$$

The image of the map is called the monodromy group of the Frobenius manifold. Denote by $\widetilde{M^{\mathrm{reg}}}$ the monodromy covering space of M^{reg}.

By the analytic continuation, we have a holomorphic map

$$\widetilde{M^{\mathrm{reg}}} \times \mathrm{Sol}(\nabla)_{p_0} \longrightarrow \mathbb{C}, \quad (\widetilde{p}, x) \mapsto x(\widetilde{p}).$$

Put $\mathbb{E} := \mathrm{Hom}_{\mathbb{C}}(\mathrm{Sol}(\nabla)_{p_0}, \mathbb{C})$.

Definition 7.

The period mapping associated to the Frobenius manifold is the holomorphic map

$$\widetilde{M^{\mathrm{reg}}} \longrightarrow \mathbb{E}, \quad \widetilde{p} \mapsto (x \mapsto x(\widetilde{p})).$$

Roughly speaking, the period mapping associated to a Frobenius manifold is "the change of coordinate" from a flat coordinate system $(t_1, \ldots, t_n)$ to a ∇-flat coordinate system $(x_1, \ldots, x_n)$.

Let $Q = \{Q_0, Q_1\}$ be a connected finite acyclic quiver, and set $Q_0 = \{1, \ldots, n\}$.

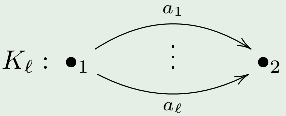
A matrix $A_Q = (a_{ij})$ of size n defined by

$$a_{ij} := 2\delta_{ij} - (q_{ij} + q_{ji}), \quad q_{ij} := \#\{i \to j \in Q_1\}, \quad \text{for } i, j \in Q_0,$$

is called the generalized Cartan matrix of Q.

Example: $n = 2$

For $Q = K_\ell$, the generalized Cartan matrix is given by



$$, \quad A_{K_\ell} = \begin{pmatrix} 2 & -\ell \\ -\ell & 2 \end{pmatrix}.$$

- If $\ell = 1$, A_{K_1} is positive definite matrix (finite type),
- If $\ell = 2$, A_{K_2} is semi-positive (affine type),
- If $\ell \geq 3$, A_{K_ℓ} is indefinite (indefinite type).

One can obtain a root system $(L, I, \Delta^{\mathrm{re}})$ associated with Q as follows:

- Define a free abelian group L by

$$L := \bigoplus_{i=1}^{n} \mathbb{Z} \cdot \alpha_i,$$

  where, α_i is a formal generator and called the simple root of $i \in Q_0$.

- Define $\mathbb{Z}$-bilinear form $I : L \times L \longrightarrow \mathbb{Z}$ by $I(\alpha_i, \alpha_j) := a_{ij}$, called the Cartan form.

- For each $i \in Q_0$, define a reflection $r_i \in \mathrm{Aut}(L, I)$ by

$$r_i(\lambda) := \lambda - I(\lambda, \alpha_i)\alpha_i, \quad \lambda \in L.$$

  Define the Weyl group W associated with Q as the group generated by reflections $W := \langle r_1, \ldots, r_n \rangle \subset \mathrm{Aut}(L, I)$.

- Define the set of real roots Δ^{re} by

$$\Delta^{\mathrm{re}} := \{w(\alpha_i) \in L \mid w \in W,\ i \in Q_0\}.$$

Define a $\mathbb{C}$-vector space $\mathfrak{h}$ by

$$\mathfrak{h} := \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{C}) \cong \mathbb{C}^n.$$

and denote the natural coupling by $\langle -, - \rangle : \mathfrak{h}^* \times \mathfrak{h} \longrightarrow \mathbb{C}$, where $\mathfrak{h}^* := L \otimes_{\mathbb{Z}} \mathbb{C}$. The Weyl group W acts on $\mathfrak{h}$ as follows:

$$\langle \alpha, w(x) \rangle := \langle w^{-1}(\alpha), x \rangle, \quad \alpha \in \mathfrak{h}^*,\ x \in \mathfrak{h},\ w \in W.$$

The element $c := r_1 r_2 \cdots r_n \in W$ is called the Coxeter element. The tuple $(L, I, \Delta^{\mathrm{re}}, c)$ is called a generalized root system.

Let $Q = \vec{\Delta}$ be a Dynkin quiver.
In this case, the Weyl group W is finite, and the Coxeter element c has finite order. The order $h \in \mathbb{Z}_{\geq 2}$ of c which is called the Coxeter number determines the (conformal) dimension of a Frobenius manifold.

Theorem 8 (Saito, Saito–Yano–Sekiguchi, Dubrovin).

There exists a unique Frobenius structure $(\eta, \circ, e, E)$ of dimension $d = 1 - \dfrac{2}{h}$ on $\mathfrak{h}/W \cong \mathbb{C}^n$ satisfying

1. *The intersection form g is determined by the Cartan matrix $A_{\vec{\Delta}}$.*
2. *There exist W-invariant homogeneous polynomials $t_1, \cdots, t_n$ such that $(t_1, \ldots, t_n)$ is a (global) flat coordinate system of the Frobenius manifold.*
3. *The Euler vector field E is given by*

$$E = \sum_{i=1}^{n} \frac{\deg t_i}{h} t_i \frac{\partial}{\partial t_i}.$$

This Frobenius structure is based on the Chevalley's Theorem;

Theorem 9 (Chevalley's Theorem).

Let

- $\alpha_i^*(=\omega_i^\vee) \in \mathfrak{h}$: *the dual (fundamental co-weight) of* $\alpha_i \in \mathfrak{h}^*$,
- $(x_1, \cdots, x_n)$: *the linear coordinate with respect to* $\{\alpha_1^*, \cdots, \alpha_n^*\}$.

We have

1. $\mathbb{C}[\mathfrak{h}]^W \subset \mathbb{C}[\mathfrak{h}] \cong \mathbb{C}[x_1, \cdots, x_n]$ *is generated by* n *homogeneous polynomials* $p_1, \cdots, p_n$ *such that*
$$h = \deg p_1 > \deg p_2 \geq \cdots \geq \deg p_{n-1} > \deg p_n = 2.$$
2. $\{\deg p_1, \ldots, \deg p_n\}$ *does not depend on the choice of* $p_1, \cdots, p_n$.
3. *The eigenvalues of the Coxeter element* c *are*
$$\exp\left(2\pi\sqrt{-1}\,\frac{\deg p_1 - 1}{h}\right), \cdots, \exp\left(2\pi\sqrt{-1}\,\frac{\deg p_n - 1}{h}\right)$$

We obtain the Frobenius structrue $(\eta, \circ, e, E)$ in Theorem 8 as follows;

- (unit vector field e) $\quad e := \dfrac{\partial}{\partial p_1}$
- (Euler vector field E)

$$E := \sum_{i=1}^{n} \frac{1}{h} x_i \frac{\partial}{\partial x_i} = \sum_{i=1}^{n} \frac{\deg p_i}{h} p_i \frac{\partial}{\partial p_i}$$

- (metric η) Let $g : \Omega^1_{\mathfrak{h}} \times \Omega^1_{\mathfrak{h}} \longrightarrow \mathcal{O}_{\mathfrak{h}}$ be a non-degenerated $\mathcal{O}_{\mathfrak{h}}$-bilinear form induced by $I : L \times L \longrightarrow \mathbb{Z}$ under the natural identification of $T^*_x \mathfrak{h} \cong \mathfrak{h}^*$, that is,

$$g(dx_i, dx_j) := I(\alpha_i, \alpha_j).$$

It induces a symmetric $\mathcal{O}_{\mathfrak{h}/W}$-bilinear form $g : \Omega^1_{\mathfrak{h}/W} \times \Omega^1_{\mathfrak{h}/W} \longrightarrow \mathcal{O}_{\mathfrak{h}/W}$. Then, we define

$$\eta := \mathrm{Lie}_e g.$$

In order to define the product structure, we need the following

Theorem 10 (Saito–Yano–Sekiguchi).

Let $\nabla\!\!\!/$ be the Levi–Civita connection with respect to η. There exists $\nabla\!\!\!/$-flat W-invariant homogeneous polynomials $t_1, \cdots, t_n$ satisfying the conditions of Chevalley's Theorem.

- (product structure $\circ$) Let ∇ be the Levi-Civita connection with respect to $g : \Omega^1_{\mathfrak{h}/W} \times \Omega^1_{\mathfrak{h}/W} \longrightarrow \mathcal{O}_{\mathfrak{h}/W}$. The product structure $\circ$ is defined by

$$C_{ij}^k := \frac{h}{\deg t_k - 1} \sum_{a=1}^{n} \eta\left(\frac{\partial}{\partial t_i}, \frac{\partial}{\partial t_a}\right) \cdot g\left(dt_a, \nabla_{\frac{\partial}{\partial t_j}} dt_k\right)$$

for $i, j, k \in Q_0$, and

$$\frac{\partial}{\partial t_i} \circ \frac{\partial}{\partial t_j} := \sum_{k=1}^{n} C_{ij}^k \frac{\partial}{\partial t_k}.$$

Define the regular subset $\mathfrak{h}^{\mathrm{reg}}$ of $\mathfrak{h}$ by

$$\mathfrak{h}^{\mathrm{reg}} := \mathfrak{h} \setminus \bigcup_{\alpha \in \Delta^{\mathrm{re}}} H_\alpha,$$

where $H_\alpha := \{x \in \mathfrak{h} \mid \langle \alpha, x \rangle = 0\}$ is the root hyperplane of $\alpha \in \Delta^{\mathrm{re}}$.

For the Frobenius manifold $(\mathfrak{h}/W, \eta, \circ, e, E)$, we can consider the regular locus of $\mathfrak{h}/W$. This is naturally isomorphic to the W-orbit space of $\mathfrak{h}^{\mathrm{reg}}$.

$$(\mathfrak{h}/W)^{\mathrm{reg}} \cong \mathfrak{h}^{\mathrm{reg}}/W$$

For an A_n-quiver, Ikeda showed that the universal covering space $\widetilde{(\mathfrak{h}/W)}^{\mathrm{reg}}$ of $(\mathfrak{h}/W)^{\mathrm{reg}}$ with the period mapping can be identified with the space of stability conditions of "the derived category of N-Calabi–Yau completion of A_n-quiver $\check{\mathcal{D}}_N(A_n)$" with the central charge map:

$$\begin{array}{ccc} \mathrm{Stab}^\circ(\check{\mathcal{D}}_N(A_n)) & \xrightarrow{\cong} & \widetilde{(\mathfrak{h}/W)}^{\mathrm{reg}} = \widetilde{\mathfrak{h}^{\mathrm{reg}}} \\ \mathcal{Z}\downarrow & \circlearrowleft & \downarrow \text{period} \\ \mathfrak{h} & \xrightarrow{\cong} & \mathbb{E} \end{array}$$

where $^\circ$ denotes a connected component.

Let $Q = K_\ell$ the ℓ–Kronecker quiver with $\ell \geq 3$.
The Coxeter element c does not have finite order. What is h in this case?

Let ρ be the spectral radius of the Coxeter element c:

$$\rho = \frac{\ell^2 - 2 + \sqrt{\ell^4 - 4\ell^2}}{2} \ (> 1).$$

The eigenvalues of c are ρ and ρ^{-1}. It can be regarded as follows:

$$\rho = \exp\left(2\pi\sqrt{-1}\frac{\log\rho}{2\pi\sqrt{-1}}\right), \quad \rho^{-1} = \exp\left(-2\pi\sqrt{-1}\frac{\log\rho}{2\pi\sqrt{-1}}\right).$$

Define a number $h \in \mathbb{C}\backslash\mathbb{R}$ by $h := \dfrac{2\pi\sqrt{-1}}{\log\rho}$ and hence we have

$$\rho = \exp\left(2\pi\sqrt{-1}\frac{2-1}{h}\right), \quad \rho^{-1} = \exp\left(2\pi\sqrt{-1}\frac{h-1}{h}\right).$$

Define the set of imaginary roots Δ_+^{im} by

$$\Delta_+^{\mathrm{im}} := \{w(\alpha) \in L \mid w \in W,\ \alpha \in L_+ \text{ s.t. } I(\alpha, \alpha_i) \leq 0,\ i = 1, 2\}.$$

and the imaginary cone $\mathcal{I} \subset \mathfrak{h}_{\mathbb{R}}^*$ by the closure of the convex hull of $\Delta_+^{\mathrm{im}} \cup \{0\}$.

Definition 11 (Ikeda).

Define an open subset $X \subset \mathfrak{h}$ by

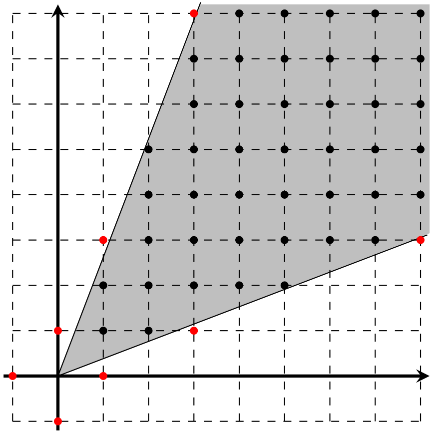
$$X := \mathfrak{h} \setminus \bigcup_{\lambda \in \mathcal{I} \setminus \{0\}} H_\lambda$$

and a regular subset $X^{\mathrm{reg}} \subset X$ by

$$X^{\mathrm{reg}} := X \setminus \bigcup_{\alpha \in \Delta^{\mathrm{re}}} H_\alpha,$$

where $H_\lambda := \{Z \in \mathfrak{h} \mid Z(\lambda) = 0\}$ for $\lambda \in \mathfrak{h}^*$.

It is known that the fundamental group is $\pi_1(X) \cong \mathbb{Z}$.



Example of the case $\ell = 3$: red dots = real roots Δ^{re}, black dots = positive imaginary roots Δ^{im}_{+}, gray shaded domain = imaginary cone $\mathcal{I}$.

The following theorem is one of the reasons why the space X is suitable:

Theorem 12 (Ikeda).

Let Q be an acyclic connected finite quiver and $\check{\mathcal{D}}_2(Q)$ "the derived category of 2-Calabi–Yau completion of Q". The natural map induced by the central charge map $\mathcal{Z}$ is a covering map

$$\mathrm{Stab}^\circ(\check{\mathcal{D}}_2(Q)) \longrightarrow X^{\mathrm{reg}}/W,$$

where $\mathrm{Stab}^\circ(\check{\mathcal{D}}_2(Q))$ is a connected component of $\mathrm{Stab}(\check{\mathcal{D}}_2(Q))$.

There exists a W-action on the universal covering space $\widetilde{X}$ such that the map $\widetilde{X} \longrightarrow X$ is W-equivariant.

Definition 13.

Define a complex analytic space $\widetilde{X}/\!\!/W$ as follows:

- *The underlying space is the quotient space $\widetilde{X}/W$ and denote by $\pi : \widetilde{X} \to \widetilde{X}/W$ the quotient map.*
- *The structure sheaf is $\mathcal{O}_{\widetilde{X}/\!\!/W} := \pi_* \mathcal{O}_{\widetilde{X}}^W$, where $\mathcal{O}_{\widetilde{X}}^W$ is the W-invariant subsheaf of $\mathcal{O}_{\widetilde{X}}$.*

Dimitrov–Katzarkov showed that $\mathrm{Stab}(\mathcal{D}^b(K_\ell)) \cong \mathbb{C} \times \mathbb{H}$ as complex manifolds, where $\mathbb{H} = \{z \in \mathbb{C} \mid |\mathrm{Im}(z)| > 0\}$.

Proposition 14.

$\widetilde{X}/\!\!/W$ is a complex manifold. Moreover, there exists an isomorphism

$$\widetilde{X}/\!\!/W \cong \mathrm{Stab}(\mathcal{D}^b(K_\ell))$$

as complex manifolds.

Theorem 15 (Ikeda-O-Shiraishi-Takahashi).

There exists a unique Frobenius structure $(\eta, \circ, e, E)$ *of dimension* $1 - \dfrac{2}{h}$ *on* $\widetilde{X}/\!\!/W$ *satisfying*

1. *The intersection form* g *is determined by the generalized Cartan matrix* A_{K_ℓ}.
2. *The functions* (t_1, t_2) *defined by*

$$t_1 = e^{hy_1} - e^{hy_2} = \text{“}x_1^h - x_2^h\text{”}, \quad t_2 = e^{y_1+y_2} = x_1 x_2$$

   are W*-invariant homogeneous and forms a flat coordinate system of the Frobenius structure, where* (y_1, y_2) *is the "natural" coordinate system on* $\widetilde{X}$.
3. *The Euler vector field* E *is given by*

$$E = t_1 \frac{\partial}{\partial t_1} + \frac{2}{h} t_2 \frac{\partial}{\partial t_2}.$$

Let $M := (\mathbb{C}^*)^n \times \mathrm{Conf}^n(\mathbb{C})$, where

$$\mathrm{Conf}^n(\mathbb{C}) = \{(p_1, \ldots, p_n) \in \mathbb{C}^n \,|\, p_i \neq p_j \ (i \neq j)\}.$$

Consider a function $F : \mathbb{C} \times M \longrightarrow \mathbb{C}$ defined by

$$F(z; \mathbf{q}, \mathbf{p}) := z + \sum_{i=1}^{n} \frac{q_i}{z - p_i}, \quad \mathbf{q} \in (\mathbb{C}^*)^n,\ \mathbf{p} \in \mathrm{Conf}^n(\mathbb{C}),$$

and $\zeta := \dfrac{dz}{z}$.

It was shown by Dubrovin in the viewpoint of the Hurwitz space that the pair (F, ζ) induces a Frobenius structure $(\eta_\zeta, \circ, e, E)$ on M.

Theorem 16 (Ikeda–O–Shiraishi–Takahashi).

Define a local coordinate system $\mathbf{t} = (t_1, \dots, t_n, t_{1^*}, \dots, t_{n^*})$ *by*

$$F(z;\mathbf{t}) = z + \frac{e^{t_{1^*}}}{z} + t_1 + \sum_{i=2}^{n} \frac{z}{z - e^{t_{i^*}}} t_i.$$

Then it forms a flat coordinate system of the Frobenius manifold $(M, \eta_\zeta, \circ, e, E)$. *Moreover, we have*

$$E = t_1\partial_1 + \sum_{i=2}^{n} t_i\partial_i + 2\partial_{1^*} + \sum_{i=2}^{n} \partial_{i^*},$$

and

$$\eta_\zeta(\partial_a, \partial_b) = \begin{cases} 1, & (b = a^*), \\ 0, & (\textit{otherwise}). \end{cases}$$

Theorem 17 (Ikeda–O–Shiraishi–Takahashi).

The Frobenius potential $\mathcal{F}$ is given by

$$\begin{aligned}\mathcal{F}(\mathbf{t}) \;&=\; \sum_{i=1}^{n}\frac{1}{2}t_i^2 t_{i^*} + \sum_{i=2}^{n} t_1 t_i t_{i^*} + e^{t_{1^*}} - \sum_{i=2}^{n} t_i e^{t_{1^*}-t_{i^*}} \\ &\quad + \sum_{\substack{2\le i,j\le n \\ i\neq j}} t_i t_j \log(e^{t_{i^*}} - e^{t_{j^*}}) + \sum_{i=2}^{n}\frac{1}{2}t_i^2\log t_i + \sum_{i=2}^{n} t_i e^{t_{i^*}}.\end{aligned}$$

By this theorem, we can calculate the intersection form g:

$$g(dt_i, dt_j) = \sum_{a,b=1}^{n} \eta^{ia}\eta^{jb} E\partial_a\partial_b\mathcal{F}.$$

Let ∇ be the second structure connection of the Frobenius manifold $(M, \eta_\zeta, \circ, e, E)$. That is, ∇ is the Levi–Civita connection with respect to the intersection form g.

Theorem 18 (Ikeda–O–Shiraishi–Takahashi).

There exists a ∇-flat coordinate system $(x_1, \dots, x_n, x_{1^*}, \dots, x_{n^*})$ such that the matrix representation of the intersection form g with respect to the basis is given by

$$(g(dx_a, dx_b)) = \begin{pmatrix} -I_{A_n}^{-1} & 0 \\ 0 & I_{A_n} \end{pmatrix},$$

where I_{A_n} is the Cartan matrix of A_n type

$$I_{A_n} = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ 0 & -1 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 2 \end{pmatrix}.$$

(Sketch of proof)

Define $p_i(\mathbf{x})$ by

$$p_i(\mathbf{x}) := \begin{cases} \mathbf{e}\Big[\sum_{j=1}^{n} \frac{j}{n+1} x_{j^*}\Big]\mathbf{e}\left[x_1\right], & i = 1, \\ \mathbf{e}\Big[\sum_{j=1}^{n} \frac{j}{n+1} x_{j^*}\Big]\mathbf{e}\left[x_i - x_{i-1}\right], & i = 2, 3, \ldots, n, \\ \mathbf{e}\Big[\sum_{j=1}^{n} \frac{j}{n+1} x_{j^*}\Big]\mathbf{e}\left[-x_n\right], & i = n+1. \end{cases}$$

Consider

$$F(z; \mathbf{x}) = \frac{\prod_{i=1}^{n+1} (z - p_i(\mathbf{x}))}{z \prod_{k=2}^{n} \Big(z - \mathbf{e}\Big[\sum_{j=k}^{n} x_{j^*}\Big]\Big)}.$$

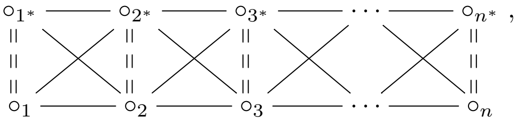
By comparing the coefficients $F(z;\mathbf{x})$ and $F(z;\mathbf{t})$ (or taking residue), we have

$$\begin{cases} t_1 = -e^{-\sum_{k=2}^{n} t_{k^*}} \displaystyle\sum_{k=1}^{n+1} \prod_{\substack{1\le l\le n+1 \\ l\ne k}} p_l(\mathbf{x}) + \sum_{i=2}^{n} e^{t_{1^*}-t_{i^*}}, \\ t_i = \dfrac{\displaystyle\prod_{l=1}^{n+1}(e^{t_{i^*}} - p_l(\mathbf{x}))}{e^{2t_{i^*}} \displaystyle\prod_{\substack{2\le k\le n \\ k\ne i}} (e^{t_{i^*}} - e^{t_{k^*}})}, \quad (i = 2, \dots, n), \\ t_{i^*} = \displaystyle\sum_{k=i}^{n} x_{k^*}, \quad (i^* = 1^*, \dots, n^*). \end{cases}$$

Hence, we obtain the statement by direct calculations. □

Let $R^{(n)}_{A_n} = (L, I, \Delta^{\mathrm{re}})$ be a root system with the signature $(n, n, 0)$ such that $R^{(n)}_{A_n}/\mathrm{rad}\, I$ is isomorphic to the root system of A_n type:

- $L := \bigoplus_{i=1}^{n} \mathbb{Z}\alpha_i \oplus \bigoplus_{i=1}^{n} \mathbb{Z}\alpha_{i^*}$ is a free $\mathbb{Z}$-module of rank $2n$,
- I is the Cartan form such that the Coxeter–Dynkin defined by I is



  where

$$
\begin{aligned}
I(\alpha_i, \alpha_j) = 2 &\iff \circ_i = = = \circ_j \\
I(\alpha_i, \alpha_j) = -1 &\iff \circ_i \text{———} \circ_j \\
I(\alpha_i, \alpha_j) = 0 &\iff \circ_i \qquad\quad \circ_j
\end{aligned}
$$

  In particular, $\mathrm{rad}\, I$ has rank n.

- $\Delta^{\mathrm{re}} = \{\alpha \in L \,|\, I(\alpha, \alpha) = 2\}$.

We expect that there exists a Frobenius manifold obtained from the above root system $R^{(n)}_{A_n}$ and it is isomorphic to $(M, \eta_\zeta, \circ, e, E)$:

$$\begin{array}{ccc} \text{(Def.)} & \longleftrightarrow & \text{(Weyl)} \\ (F : \mathbb{C} \times M \longrightarrow \mathbb{C}, \zeta) & & (R^{(n)}_{A_n} = (L, I, \Delta^{\mathrm{re}}), c) \end{array}$$

Dubrovin–Zhang proved it for $n = 1$. In this case, to get a Frobenius manifold, we consider the extended affine Weyl group $\widehat{W}$ and an extension $\widehat{\mathfrak{h}} := \mathfrak{h} \times \mathbb{C}$ of the Cartan subalgebra $\mathfrak{h}$.

Based on the case of $n = 1$, we are trying to construct the "n-extended" affine Weyl group $\widehat{W}$ and the "n-extension" $\widehat{\mathfrak{h}}$ of $\mathfrak{h}$ such that $\widehat{\mathfrak{h}}/\widehat{W} \cong M$ and it has a Frobenius structure.

Thank you for your attention !